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On the Higher Singularities of Plane Curves.

BY CHARLOTTE ANGAS SCOTT.

1. The question of the analysis of Higher Singularities is one whose importance has been recognized from the time of Cramer. Two of the principal methods of dealing with it—by expansions, and by successive quadric transformations—are used in his "Analyse des Lignes Courbes"; the explanation of the principles involved, and the development of the theory, belong to the present day. The memoirs to which reference is made in this paper are those by Cayley, "On the Higher Singularities of a Plane Curve," 1866 (Quarterly Journal, VII, 212), and H. J. Smith, "On the Higher Singularities of Plane Curves," 1873–6 (Proc. of the London Math. Soc. VI, 153), and the series of papers by Brill and Nöther in the Mathematische Annalen, IX (1876), XVI (1880), XXIII (1884), etc.

2. The question consists of two principal parts. In the first place, in the case of any composite singularity the methods enable us to assign numbers δ, $\varkappa$, for the nodes and cusps that would produce the same deficiency and the same reduction in the class of the curve as the singularity in question. But these being determined, there is then the geometrical side to be considered. Is there any geometrical reality corresponding to this algebraic symbol? Do the numbers δ, $\varkappa$, express actual facts, or are they a conventional representation of the point to satisfy Plücker's equations? i. e. having determined that a certain compound singularity is equivalent to δ nodes, $\varkappa$ cusps, etc., is there a penultimate form in which these singularities exist indefinitely near together? This question is to a certain extent resolved by Cramer, in special examples, by means of the transformation $y = vx$;* he shows, for instance, that certain singularities with coin-

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* It is noticeable that Newton uses this transformation, $y = xy$, in his Enumeratio Lin. Ter. Ord., calling the curve so obtained a hyperbolism of the original curve.

cident tangents occur as the final form of singularities with distinct tangents, involving a number of evanescent loops; the ordinary cusp thus presents itself as a node with an evanescent loop. Taking, e. g. (p. 636) the curves

$$x^5 = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4, \tag{1}$$

Cramer applies the transformation ($y = ux$)

$$x = x_1, \ y = x_1y_1.$$

This gives us a new curve,

$$x_1 = a + by_1 + cy_1^2 + dy_1^3 + ey_1^4; \tag{2}$$

both curves being referred to the same axes we have a simple construction for determining corresponding points. The intersections of (2) with the axis of y determine the passages of (1) through the origin; an arc of (2) cut off by the axis of y corresponds to a loop of (1) closed at the origin. If, e. g., all four roots of the equation $a + bz + cz^2 + dz^3 + ez^4 = 0$ coincide, (2) has a 'serpentement' presenting the appearance of ordinary contact; (1) has at the origin a point that presents the appearance of a simple cusp; but the cusp contains three vanishing loops, and is really a quadruple point. Again, on p. 638, Cramer uses a higher transformation

$$x = x_2y_2, \ y = x_2y_2^2,$$

by means of which he reduces

$$x^6 + ax^4y + cx^2y^2 + dxy^3 + ey^4 = 0 \tag{3}$$

to depend on the conic

$$x_2^2 + ax_2 + c + dy_2 + ey_2^2 = 0. \tag{4}$$

This transformation is of course equivalent to the two

$$x = x_1, \ y = x_1y_1; \ x_1 = x_2y_2, \ y_1 = y_2;$$

but Cramer uses it in the form first given. Intersections of (4) with either axis indicate passages of (3) through the origin, the interpretation differing somewhat for the two axes; and by considering the different possible varieties and positions of the conic (4), Cramer obtains all the varieties of the quadruple point on (3).

3. These same two transformations (i. $y = vx$, ii. $x = vy$) are used by Nöther in his paper "Ueber die singulären Werthsysteme einer algebraischen Function," etc. (Math. Ann. IX, 166), which is devoted to the development of

the analytical theory; his conclusion is that any compound point-singularity, a multiple point of order k, is made up of a simple k-point, with, in directions indicated by coincident tangents, other multiple points of orders $\not> k$; with a number V of branch points, which with the same number of double points form V cusps; so that for the resolution of the singularity we have

$$\delta = \sum \frac{k_1(k_1 - 1)}{2} - V; \quad \varkappa = V.$$

4. This theory is presented by Nöther as a purely analytical one, involving no dependence on geometrical ideas even when geometrical terms are used. But by this exclusion of geometry the investigation does not gain in rigor, as regards the analysis of higher singularities of algebraic curves; and it fails to show very clearly the existence of the various elements of the compound singularity. The actual significance of the transformation and resolution appears at once when we make use of the geometrical method of Quadric Inversion, described by T. A. Hirst in the Proceedings of the Royal Society, March, 1865. For convenience and completeness I shall reproduce, with obvious extensions, such of Dr. Hirst's results as are necessary for the main object of this paper, the Graphical Analysis of Higher Singularities. There is of course no occasion for this purpose to recognize any non-projective distinctions.

5. We have a fixed origin O, and a fixed base conic. Points collinear with regard to O, conjugate with regard to the conic, are said to be *inverse*. If OI, OJ are the tangents from O to the conic, the inverse of a straight line is a conic through OIJ; the correspondence is therefore a 1 . 1 quadric correspondence, with O, I, J, as fundamental points, and we have different cases, depending on the relative position of O, I, J; (1) if the conic be a proper conic, and O not on it, the three fundamental points are distinct; (2) the conic may be a line pair, and O not on it, which implies the coincidence of I, J; (3) the conic being a proper conic, O may be on it, in which case I, O, J, are consecutive points on a curve of finite curvature; or finally (4) the conic may degenerate, and O may be on it, this happens when OIJ are collinear; this case need not be considered, as it is simply 'harmonic projection. It is convenient to call the polar of O with regard to the conic the *base line*.

If the curve to be inverted passes through any of the fundamental points O, I, J, then the fundamental lines IJ, OI, OJ, will present themselves in the inverse; they are, however, to be rejected, and the residuum is to be counted as the *proper* inverse. Thus, e. g., the inverse of a line OP through O appears as $OP \cdot IJ$. Rejecting IJ, we have: the inverse of a line through O is the line itself; and similarly, if IJ are distinct, and A a point on the base conic, the inverse of IA is JA. This gives a convenient construction for the inverse P' of a point P; let IP cut the base conic in A, then JA meets OP in P'.

6. From the definition it appears at once that the inverse of any point on the base line is the origin, and that consequently k intersections of a curve with the base line give, on the inverse, a k-point at the origin. Similarly, in case (1) for the relation of I to OI, and of J to OJ. Thus a curve of order n, with points of multiplicity i, j, k at I, J, O, gives a curve of order n', with points of multiplicity i', j', k'; where

$$\begin{aligned} n' &= 2n - i - j - k, \\ i' &= n - i - k, \\ j' &= n - j - k, \\ k' &= n - i - j. \end{aligned}$$

If however we take IJ coincident (case (2)), then an intersection with OI gives a branch touching the base line; thus corresponding to points on OI we get a number of branches with contact at I. And if we take IOJ as in case (3), we get branches having three consecutive points common.

Thus the inverse of a conic *not* passing through the fundamental points is

(i) a quartic with dps at O, I, J;
(ii) a quartic with a dp at O, and a tacnode at I;
(iii) a quartic with an oscnode at O.

7. The formulæ of transformation are obtained at once; for the three cases we get

(1) $x : y : z = x'z' : y'z' : x'y'$;
(2) $x : y : z = x'z' : y'z' : x'^2$;
(3) $x : y : z = x'^2 : x'y' : -(x'z' + 2my'^2)$;

to a projection of one or other of which, as is well known, the most general $1 \cdot 1$ quadric transformation can be reduced.

8. Applying the method to the analysis of higher singularities, we notice in the first place that a multiple point of any order, with a number of coincident

or distinct tangents, gives in the inverse, if not on a fundamental line, a multiple point of the same nature. Again, since a straight line not passing through a fundamental point becomes a conic, we see that inflexions may be gained or lost; and similarly for double tangents. The method, then, is *directly* applicable to the consideration of singularities *only as point-singularities.*

9. Attending only to cases (1), (2), the base line does not pass through the origin. Let a branch cut the base line in Z; to study the inverse it is convenient to mark off and number corresponding divisions (Fig. (1)).

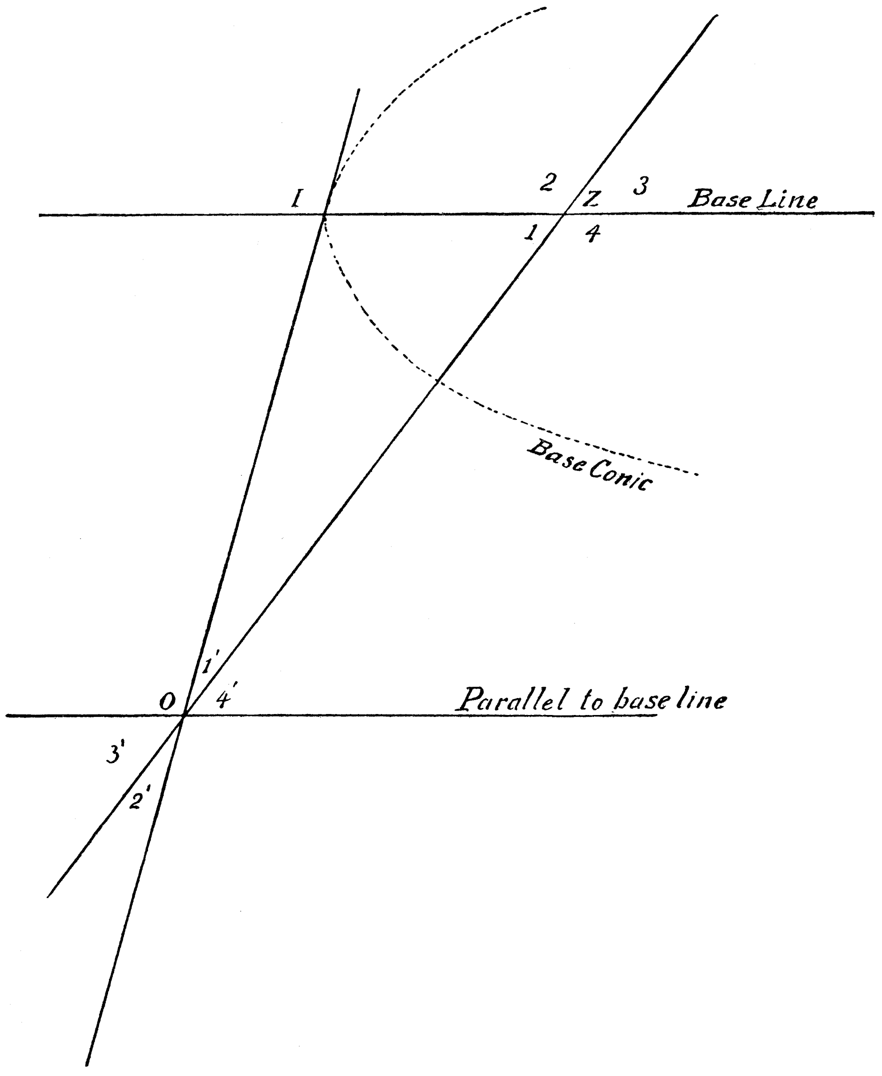

FIG. 1.

Let the branch cutting the base line at Z pass from 1 to 3; its inverse has therefore to pass from $1'$ to $3'$ through O, and so describes a branch through O, having OZ as the tangent there. If, however, the branch through Z has OZ as the tangent, it passes from 1 to 2, and its inverse, passing from $1'$ to $2'$, and touching OZ at O, has an inflexion there; and in general a branch having with OZ at Z contact of order n gives a branch having with OZ at O contact of order $n+1$.

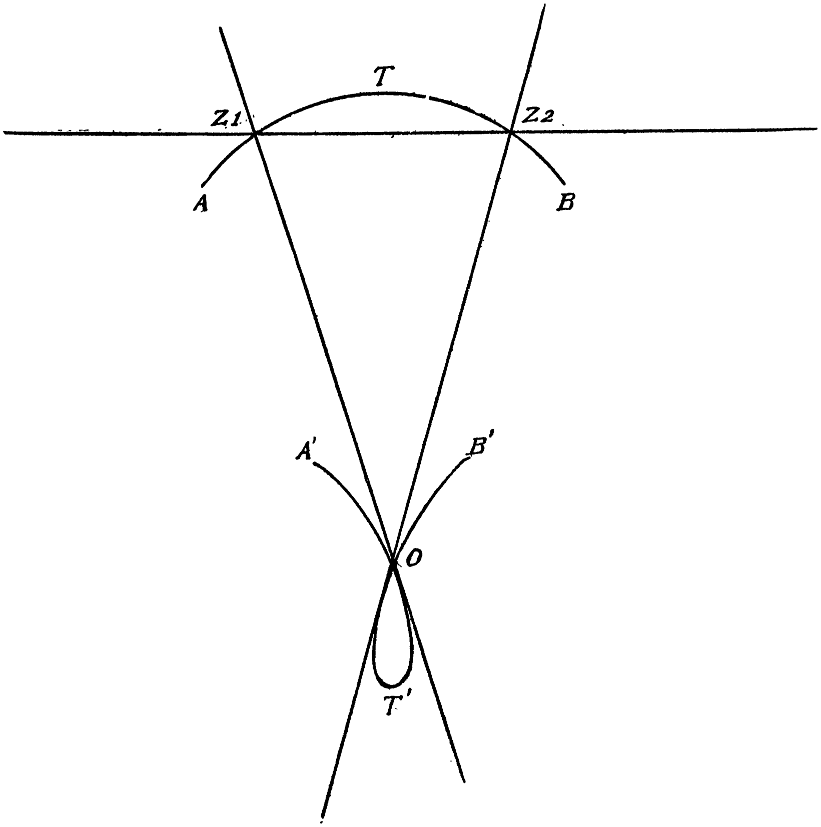

FIG. 2.

10. Next let the curve cut the base line in two points $Z_1 Z_2$ (1) on the same branch. The inverse of the arc $Z_1 T Z_2$ (Fig. (2)) is the loop $OT'O$, T' being inverse to T. If then the points $Z_1 Z_2$ close up, the arc included vanishing, so that we have ordinary contact with the base line, the loop $OT'O$ closes up, and is finally evanescent, with coincident tangents; we have then, corresponding to a branch in 1, 4 with contact at Z, a branch in $1'$, $4'$, having contact with OZ at O; we

have, that is, a cusp (Fig. (3)). The simplest cusp, then, presents itself as an evanescent loop with a node.

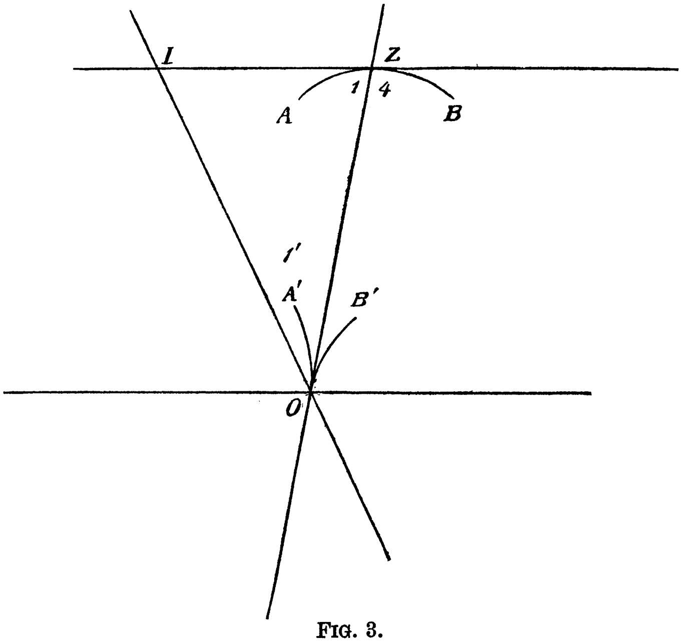

FIG. 3.

(2) But the points Z_1Z_2 need not be on the same branch; they may coincide owing to the presence of a double point at Z. In this case we have a tacnode at O. To see how this arises, consider the penultimate form, with the double point N not on the base line, but indefinitely near to it (Fig. (4)). We have

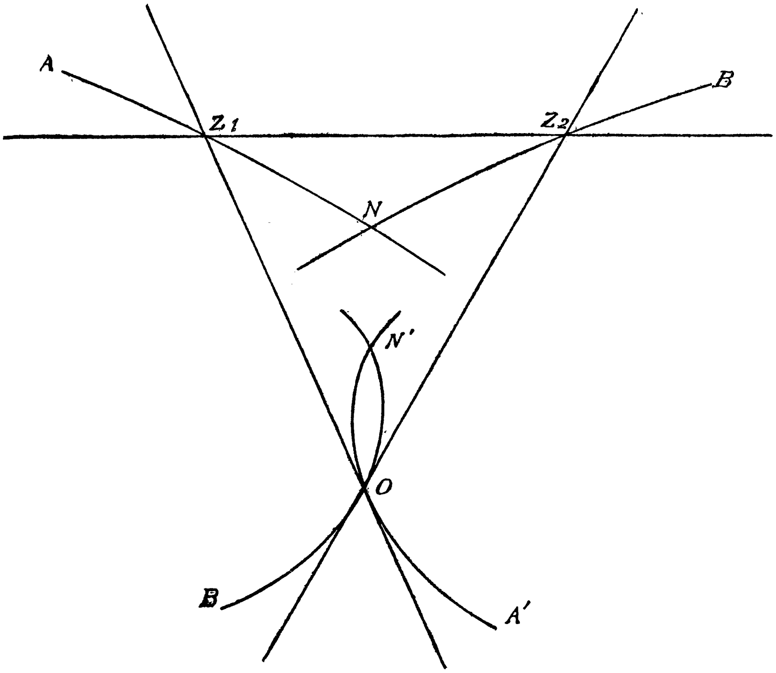

FIG. 4.

then in the inverse a node O, due to Z_1Z_2, and a node N', due to N. Thus we get a tacnode, Fig. (5), with OZ as tangent. The double point at O, then, on inversion, sheds one double point, and preserves the other.

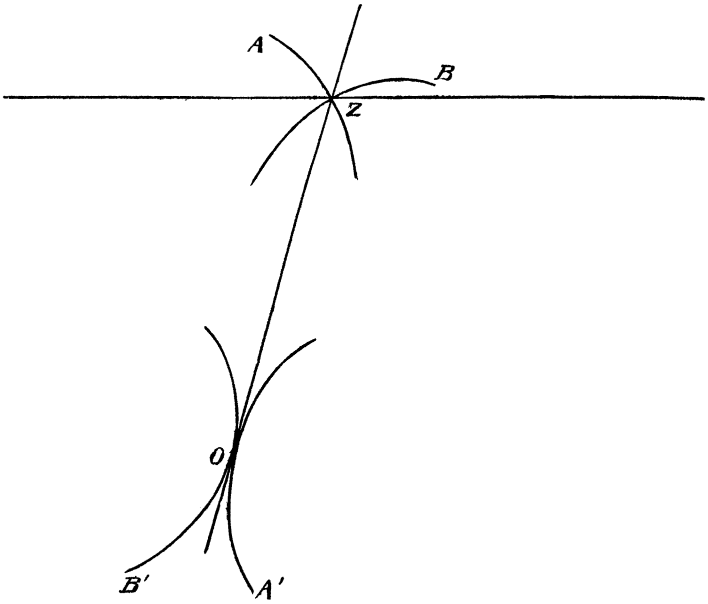

FIG. 5.

(3) Again, the points Z_1Z_2, being on the same branch, may coincide on account of a cusp. Now we already know that a cusp arises from an evanescent loop with a node; if then the cuspidal tangent does not pass through O, we have the resolution as in Figs. (6), (7), the cusp giving rise to a ramphoid cusp, whose

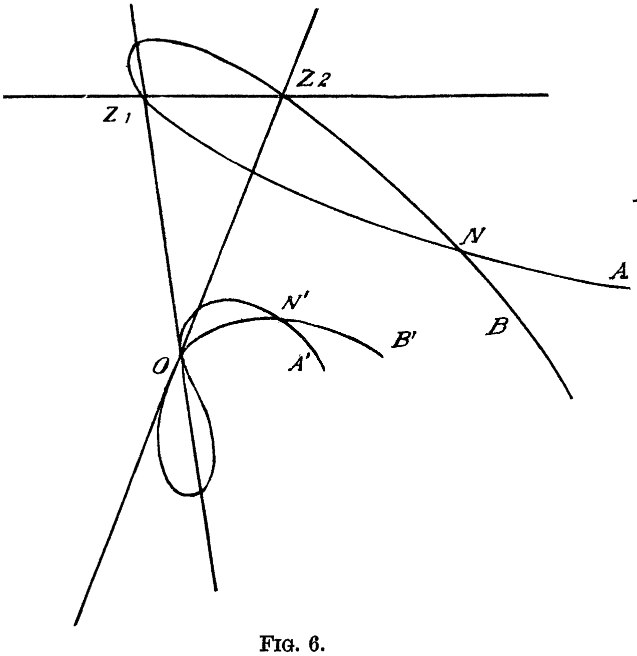

FIG. 6.

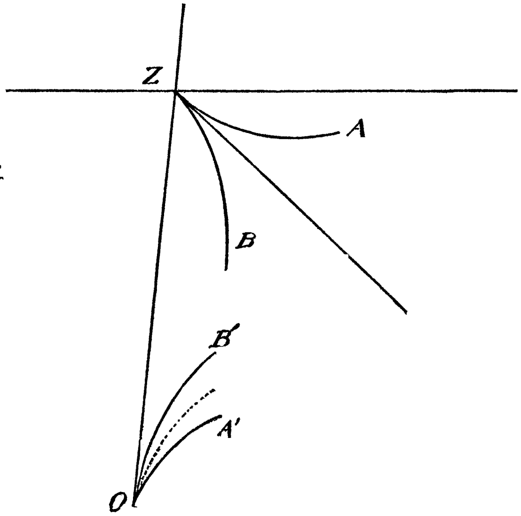

FIG. 7.

resolution into two nodes and an evanescent loop is here shown. It may be noticed, too, that the presence of the inflexion in the ramphoid cusp is shown by the one tangent from O to the evanescent loop NZ_1Z_2. If, however, the cuspidal tangent passes through O, it inverts, not into the conic of curvature as in Fig. (7), but into a straight line, and the inverse of the ceratoid cusp is a ceratoid cusp in appearance, but it is equivalent to a simple cusp with an extra node, i. e. to $y^2 = x^5$ (Figs. (8), (9)), and involves two inflexions, indicated by there being two tangents from O to the loop NZ_1Z_2.

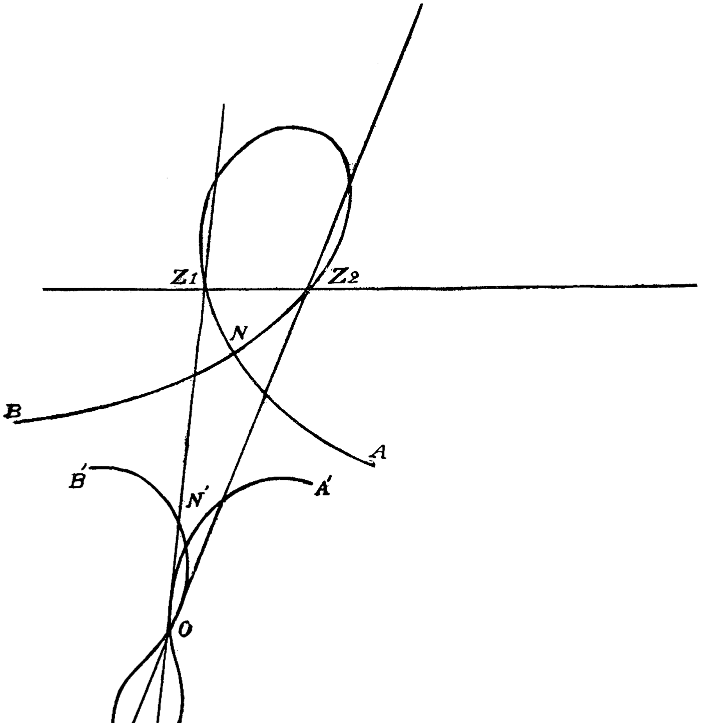

FIG. 8.

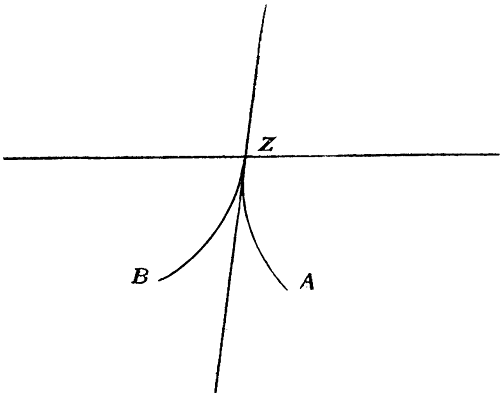

FIG. 9.

11. If now we have *three* coincident points on the base line, we get at O a triple point, differing according to the cause of the threefold coincidence. This may be (1) an inflexion. Resolving this in the recognized way, we see that the

invisible singularity resulting is really (Figs. (10), (11)) a triple point with two evanescent loops, i. e. it is equivalent to one node and two cusps (Cramer, p. 610).

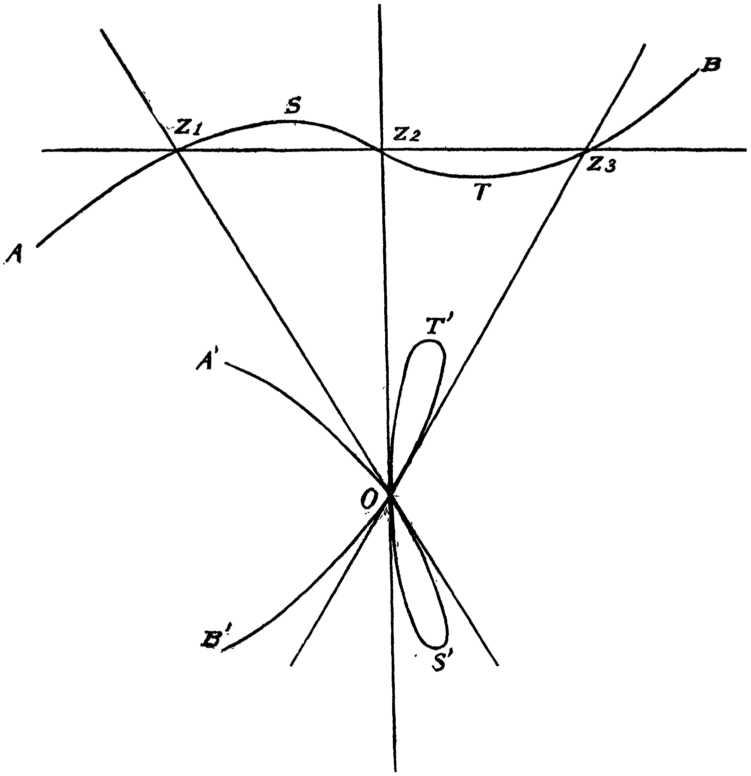

FIG. 10.

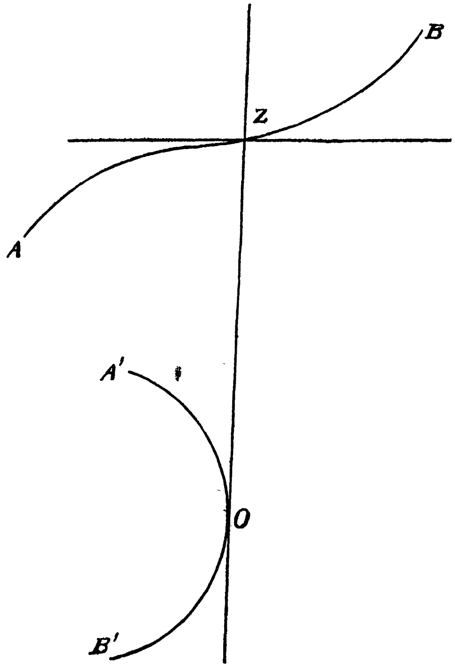

FIG. 11.

(2) There may be on the base line a node, the base line being a tangent; displacing this slightly, we see that the inverse has a cusp with an ordinary branch passing through it; the singularity is a triple point O, with a neighboring double point N', and it contains one cusp (Figs. (12), (13)).

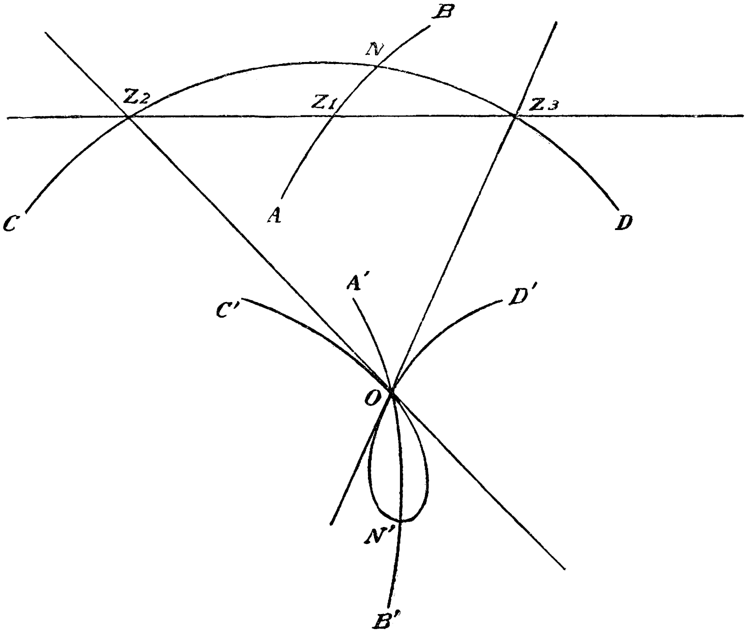

FIG. 12.

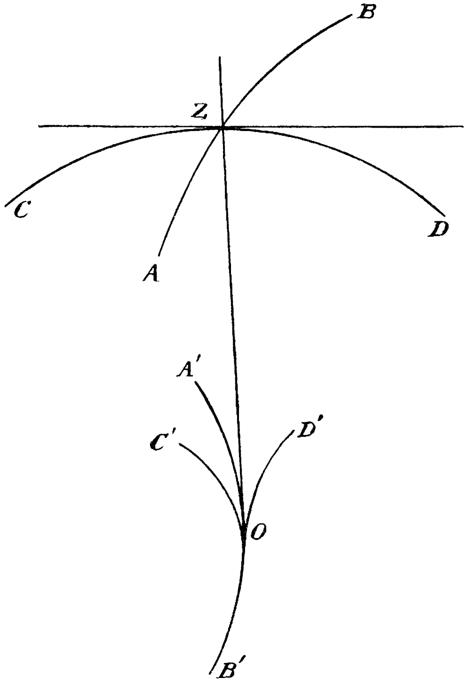

FIG. 13.

(3) There may be a cusp with the base line as tangent. Resolving this (Figs. (14), (15)) we have two evanescent loops, combined with a triple point O and a double point N'. The singularity, then, while presenting the appearance of an inflexion, gives $\delta + \varkappa = 4$, $\varkappa = 2$.

Displacing the components of the triple point O, we have Fig. (16).

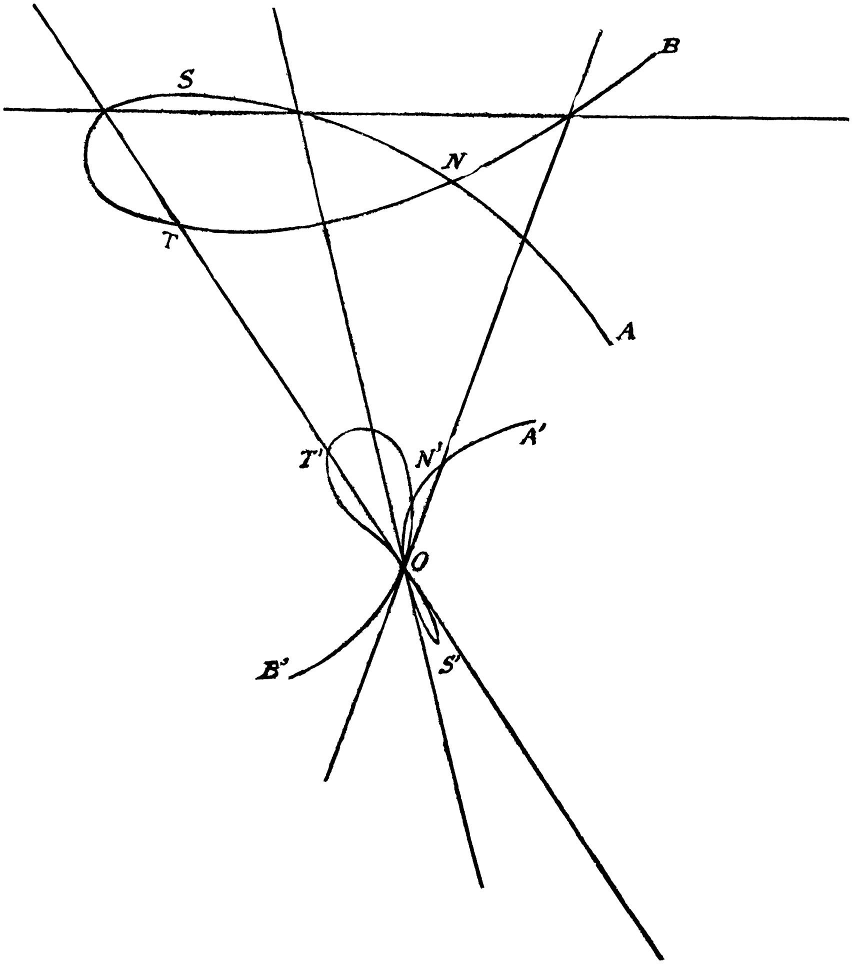

FIG. 14.

(4) If the threefold coincidence on the base line is due to a triple point, this presents itself as a triple point in the immediate neighborhood of O; we have then two consecutive triple points.

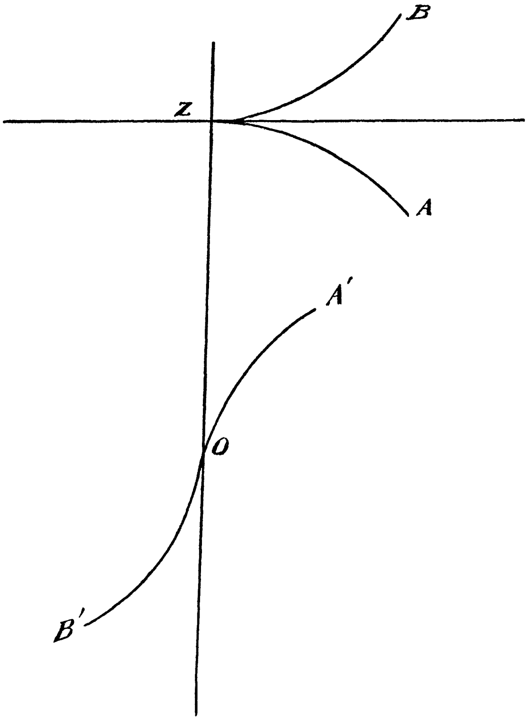

FIG. 15.

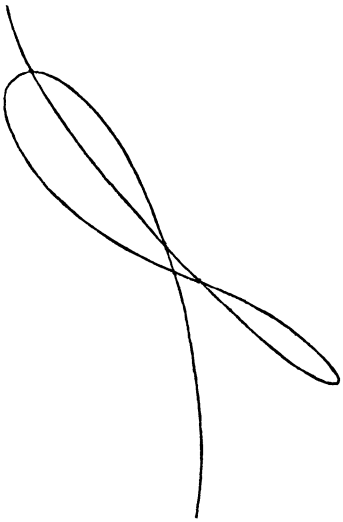

FIG. 16.

12. Reversing the process, we see that a singularity at O, when inverted, sheds the multiple point of highest order k, this being represented by k points on the base line; some or all of these will coincide if the singularity has coincident tangents, and for each such coincidence we have on the base line a multiple point of order k' $(k' \ngtr k)$. The original singularity of order k is therefore equivalent to a simple k-point, together with singularities of orders k', k'', etc. $(k' \leqq k)$. Similarly in dealing with the intersections of two curves, at a point which is an i-point on one, a k-point on the other: the obvious number of intersections is ik; but if any of the tangents are the same for the two curves, there will be neighboring intersections. To detect these, we clear away the ik already found, by means of a single inversion (Nöther, Math. Ann., XXIII, 323), and the next group of intersections is thus exposed.

13. Returning to the singularity on the one curve, let the order $\geqq k$, and let there be k coincident tangents; we have then on the base line k coincident points.

We have, that is, a singularity of order k', $k' \not> k$. If $k' < k$, the base line is a tangent to one or more branches, and therefore a certain number of evanescent loops present themselves, indicating in the resolved singularity the presence of a certain number of cusps.

These cusps necessarily present themselves when a k-point has to be made by l branches, $l < k$. Thus, e. g., if one branch has to make a triple point, it must turn back twice, and there are two cusps. In general, for a single branch to make an α-point, the inverse—also a single branch—must meet the base line in α points, giving contact of order $\alpha - 1$, and we have therefore $\varkappa = \alpha - 1$ (H. J. Smith, L. M. Soc., VI, 161, 170).

14. The geometry for the analysis of a singularity is the same whether IJ are distinct or coincident. For special lines of reasoning it may be convenient to take IJ separate, e. g. in H. J. Smith's paper the curves are generated by two pencils whose vertices may be conveniently taken for I, J. But in transformation it is essential that the branch to be inverted shall *not* touch OI (or OJ); and yet the transformations are more conveniently followed if the tangent is a side of the coördinate triangle. Thus the second form of the transformation, that with IJ coincident, is the most convenient for the analysis of singularities.

15. The formulæ are

$$x : y : z = x'z' : y'z' : x'^2,$$

the singularity being at xy with $y = 0$ as the tangent, and the inverse at $y'z'$. If then we use Cartesian coördinates x, y, 1, and write $x' : y' : z' = 1 : y_1 : x_1$, the singularity is at xy and its inverse at x_1y_1; the formulæ are $x = x_1$, $y = x_1y_1$, which is the Cramer-Nöther transformation. Here $y = 0$ (the same line as $y_1 = 0$) is a tangent; $x = 0$ is any line through the singular point, inclined at a finite angle to the tangent considered—call this the *axis*; $x_1 = 0$ is the third side of the fundamental triangle, the *base*. Thus the singularity is inverted along a tangent, from its intersection with the axis to its intersection with the base.

The process simply splits up the compound singularity into a number of simple singularities; i. e. it shows how the singularity with coincident tangents is made up of singularities with distinct tangents. No application of the method of quadric inversion will replace a k-point by the equivalent $\frac{1}{2}k(k-1)$ nodes; the most that it can do is to separate the multiple points (Nöther), and replace the cusps by evanescent loops. The simple k-point may then, however, be further resolved by displacement (e. g. Brill, Math. Ann., XVI).

16. As an example, take the curve

$$x^4y^3 = z^7,$$

which has compound singularities at xz, yz.

$$x^4 = z^7$$

gives as first inverse $$x_1^4 = z_1^3.$$

Here the base is the tangent at a triple point. Inverting along this tangent, we get for the second inverse

$$x_2 = z_2^3;$$

so the point $x_1^4 = z_1^3$ is that illustrated in Figs. (10), (11), a triple point with two evanescent loops. Inverting this, we get the original point, a quadruple point with a neighboring triple point,* these implying 9 nodes and cusps. The presence of the three loops shows that there are three cusps; we have therefore

$$\delta = 6, \ \varkappa = 3$$

(Figs. (17), (18)).

The other singularity, $y^3 = z^7$, gives as the first inverse $y_1^3 = z_1^4$, the same point, but with the base line *not* a tangent. Figs. (19), (20) show that the original singularity is made up of two triple points, involving two cusps: i. e. $\delta = 4$, $\varkappa = 2$.

The curve $x^4y^3 = z^7$, which presents the appearance of Fig. (21), is therefore, in its resolved form, represented by Fig. (22).†

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*In Fig. 17, the elements of the triple point are drawn separated by the base line; in the original the triple point therefore presents itself as three nodes.

† That the numbers obtained by this process agree with those obtained by means of the expansions appears at follows. Let the branch be made up of k partial branches, giving expansions

$$y = x^{\alpha_1} + x^{\beta_1} + \ldots.$$
$$y = x^{\alpha_2} + x^{\beta_2} + \ldots.;$$

the transformation $y = xy$ reduces these to

$$y = x^{\alpha_1 - 1} + x^{\beta_1 - 1} + \text{ etc.}$$

Every exponent, and consequently the exponent at which any two expansions separate, is reduced by unity; i. e. since there are k expansions, twice the number of intersections is reduced by $k(k-1)$, and therefore the singularity sheds a point of order k, etc.

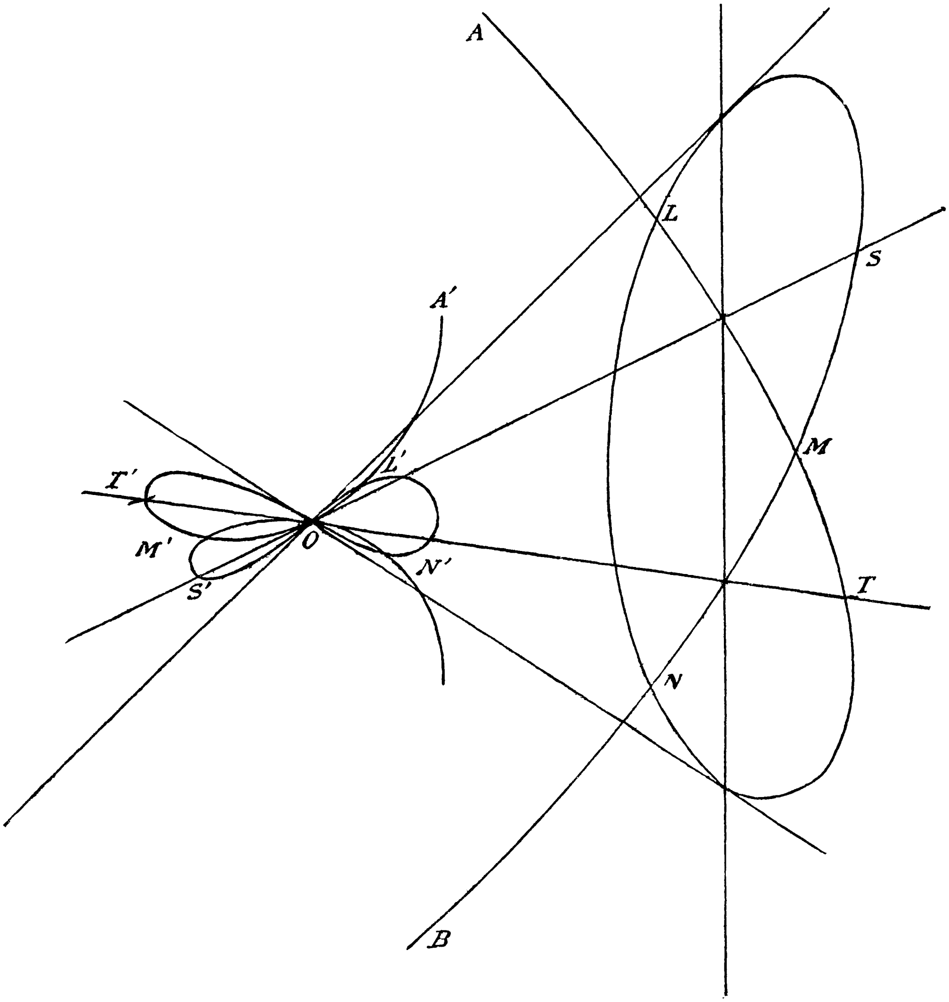

FIG. 17.

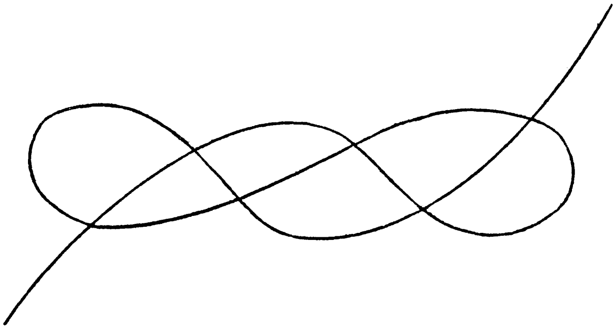

FIG. 20.

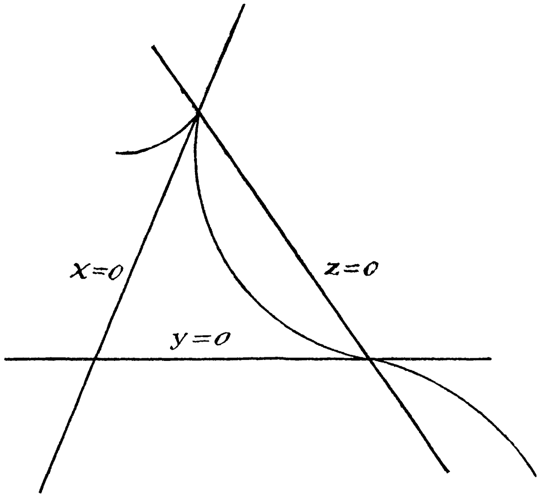

FIG. 21.

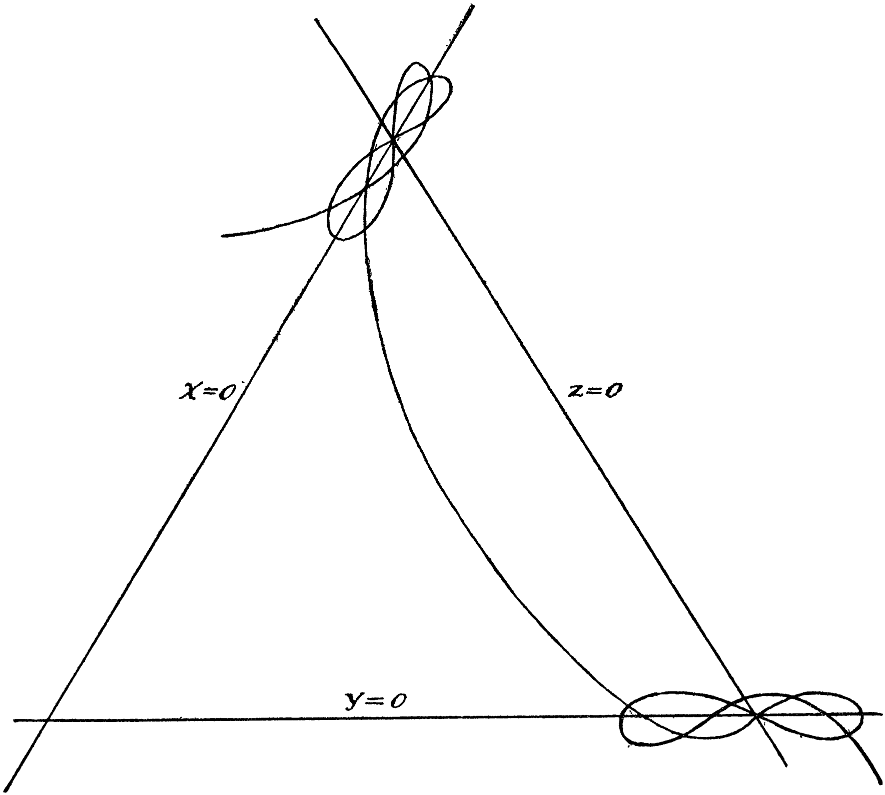

Fig. 22.

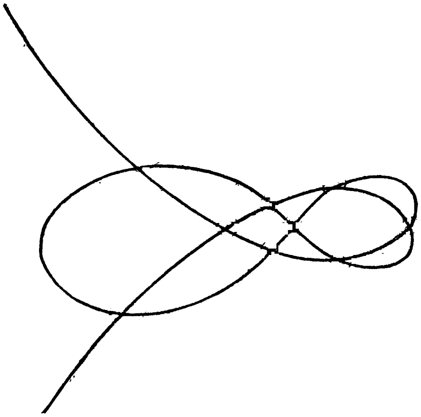

Fig. 18.

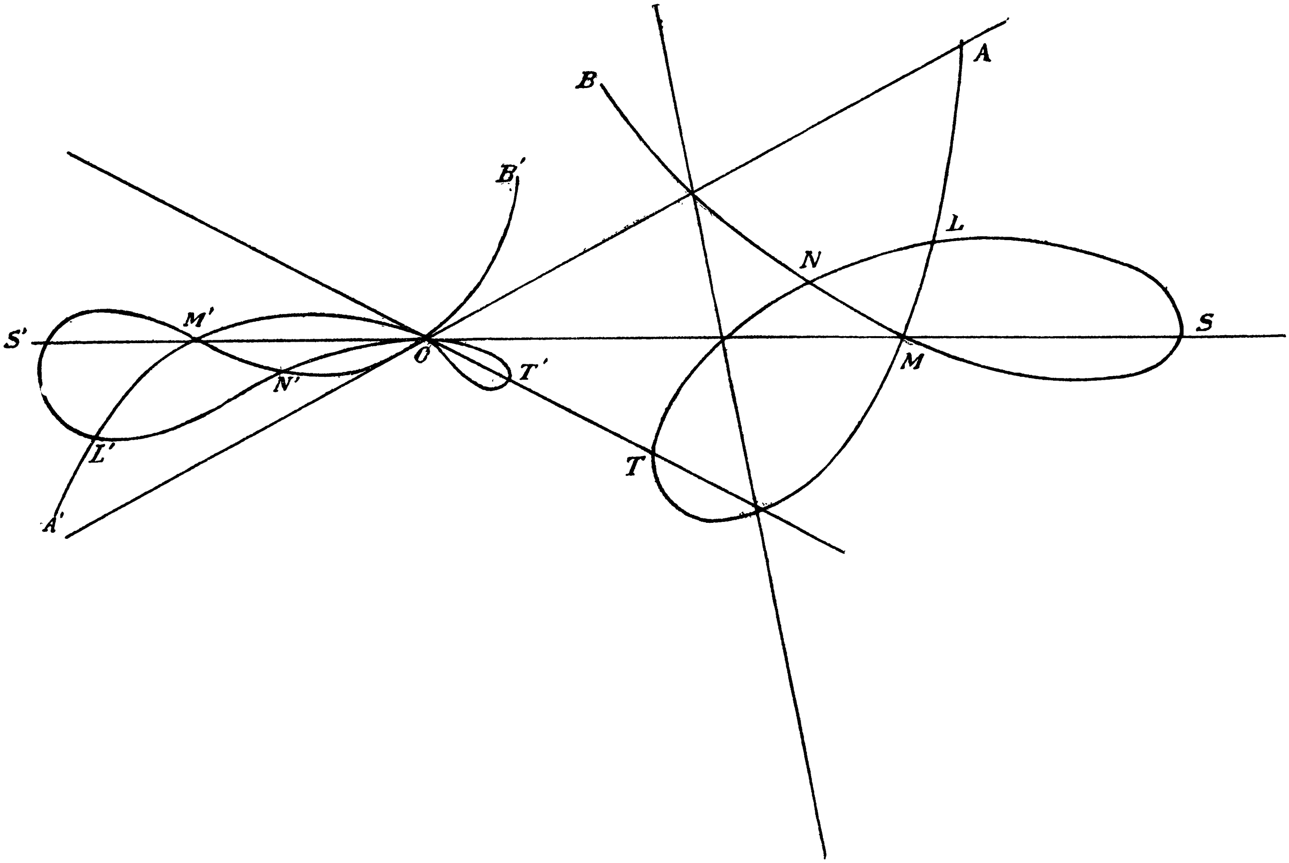

Fig. 19.

17. As a further example, take

$$y^6 - 2x^4y^3 + x^8 - 9x^5y^4 = 0 \qquad (1)$$

a curve having at the origin a singularity affording the expansions of Professor Cayley's paper in Vol. VII of the Quarterly Journal, and giving $\delta = 16$, $\varkappa = 5$.

The inverses are

$$y_1^6 - 2x_1y_1^3 + x_1^2 - 9x_1^3y_1^4 = 0 \qquad (2)$$

$$y_2^4 - 2x_2y_2^2 + x_2^2 - 9x_2^3y_2^5 = 0 \qquad (3)$$

$$y_3^2 - 2x_3y_3 + x_3^2 - 9x_3^3y_3^6 = 0 \qquad (4)$$

In this, write $x_3 - y_3 = u_3$, and it becomes

$$u_3^2 - 9y_3^6(u_3 + y_3)^3 = 0.$$

It is convenient to start afresh at this point and analyse independently the singularity

$$y^2 - 9x^6(x + y)^3 = 0 \qquad (5)$$

We get

$$y_1^2 - 9x_1^7(1 + y_1)^3 = 0 \qquad (6)$$

$$y_2^2 - 9x_2^5(1 + x_2y_2)^3 = 0 \qquad (7)$$

$$y_3^2 - 9x_3^3(1 + x_3^2y_3)^3 = 0 \qquad (8)$$

$$y_4^2 - 9x_4(1 + x_4^3y_4)^3 = 0 \qquad (9)$$

giving Fig. (23), which shows the nature of the singularity on (5).

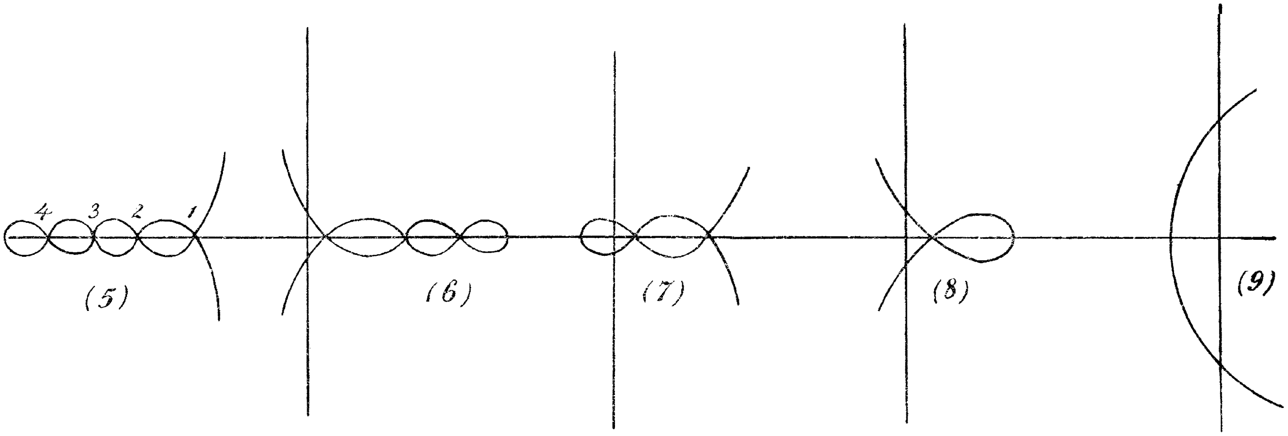

FIG. 23.

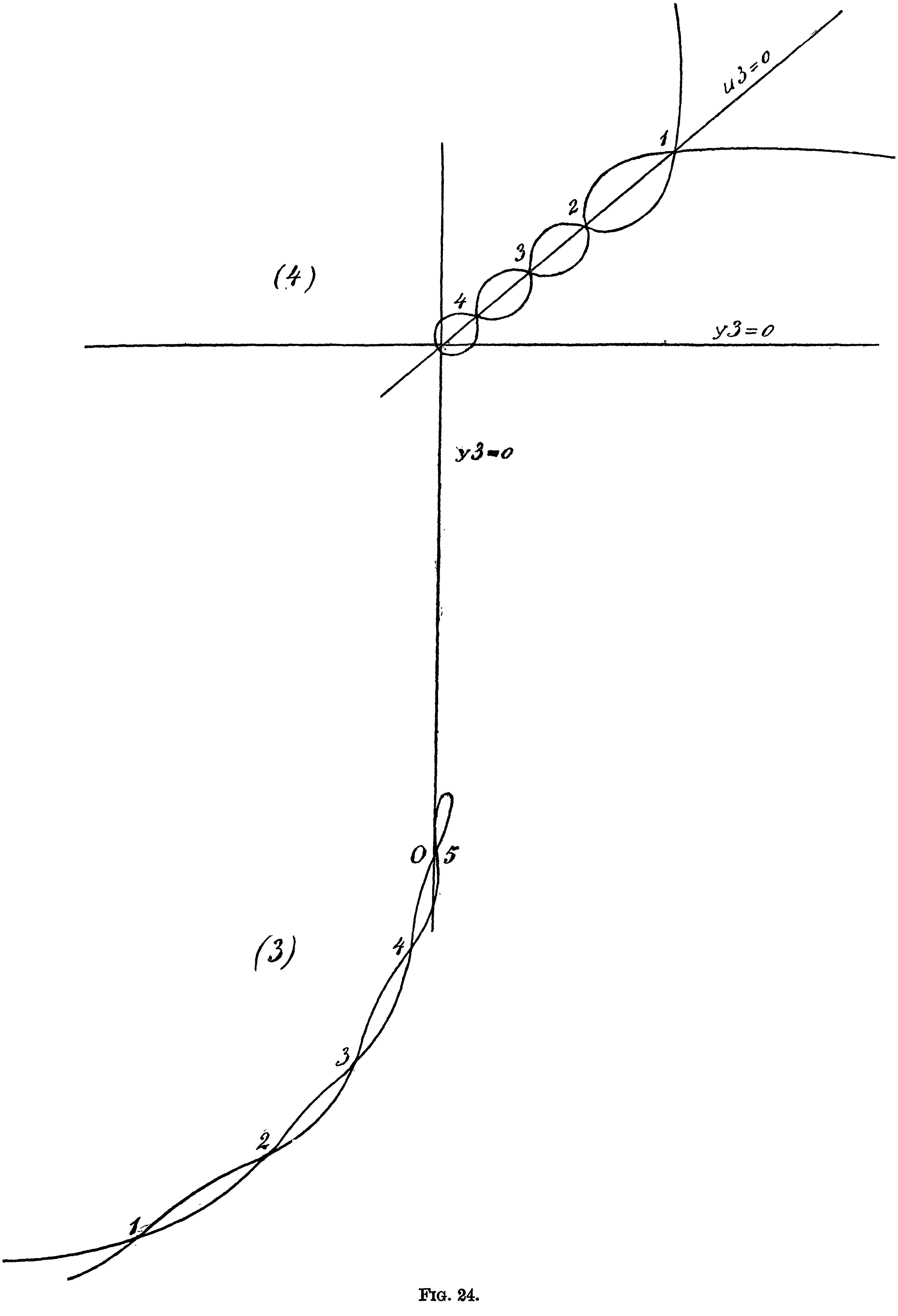

FIG. 24.

The transition from (5) (i. e. (4)) to (3) is shown in Fig. (24), the straight line on which lie the four nodes being now curved into a conic.*

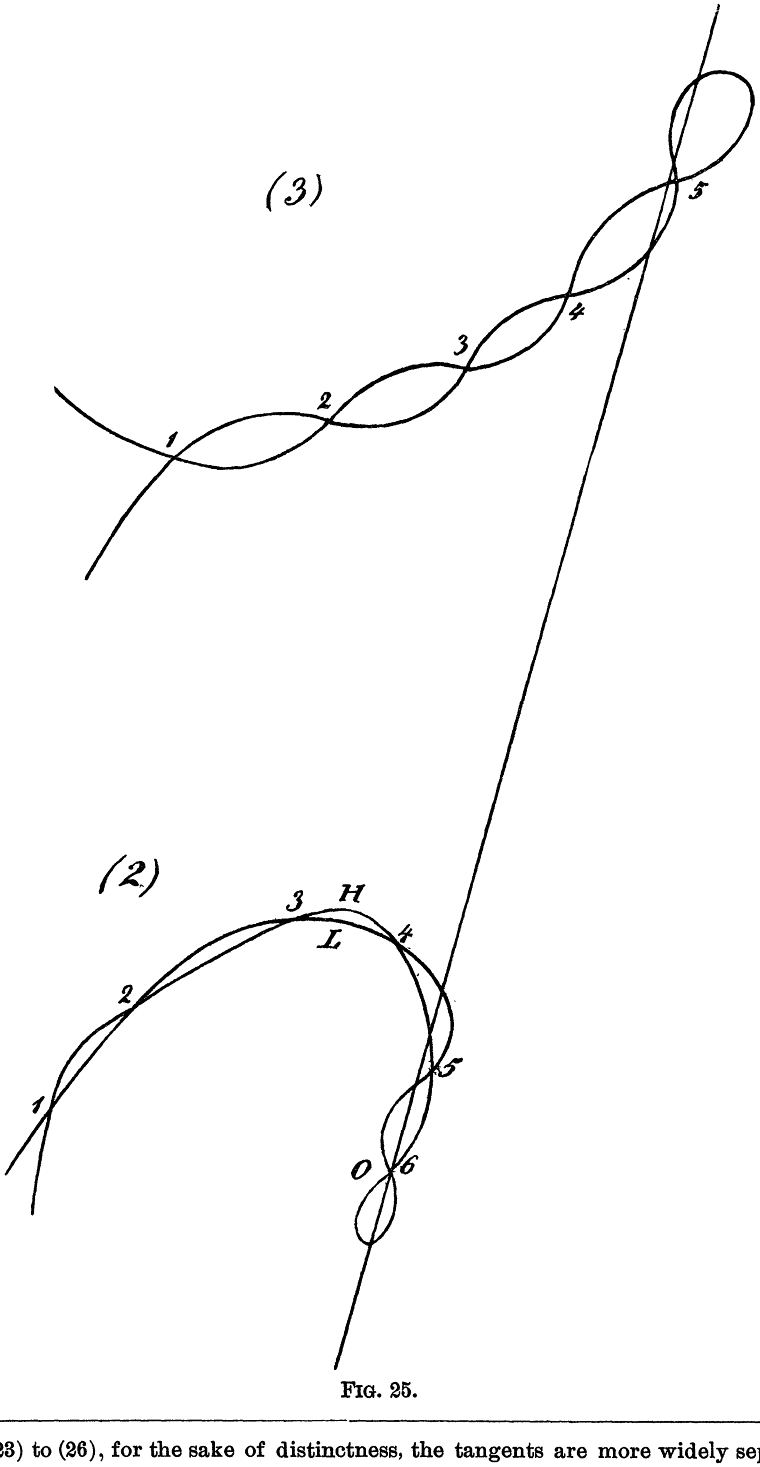

FIG. 25.

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* In Figs. (23) to (26), for the sake of distinctness, the tangents are more widely separated than is justified by the construction; and the two branches that cross and recross at 1.2.3. etc. are drawn with inflexions, as the scale is too small to keep them distinct in any other way.

The line $x_2 = 0$ (i. e. $x_3 = 0$) meets (3) in *four* points; the base line meets it in *two*. The transition from (3) to (2) introduces therefore one more node, so the first inverse of the given curve has six dps, consecutive points on some curve; one of the dps being a cusp (Fig. (25)).

In passing from (2) to (1) the base line is $x_1 = 0$, meeting the curve in six points; this finally introduces the sextic point, as in Fig. (26).

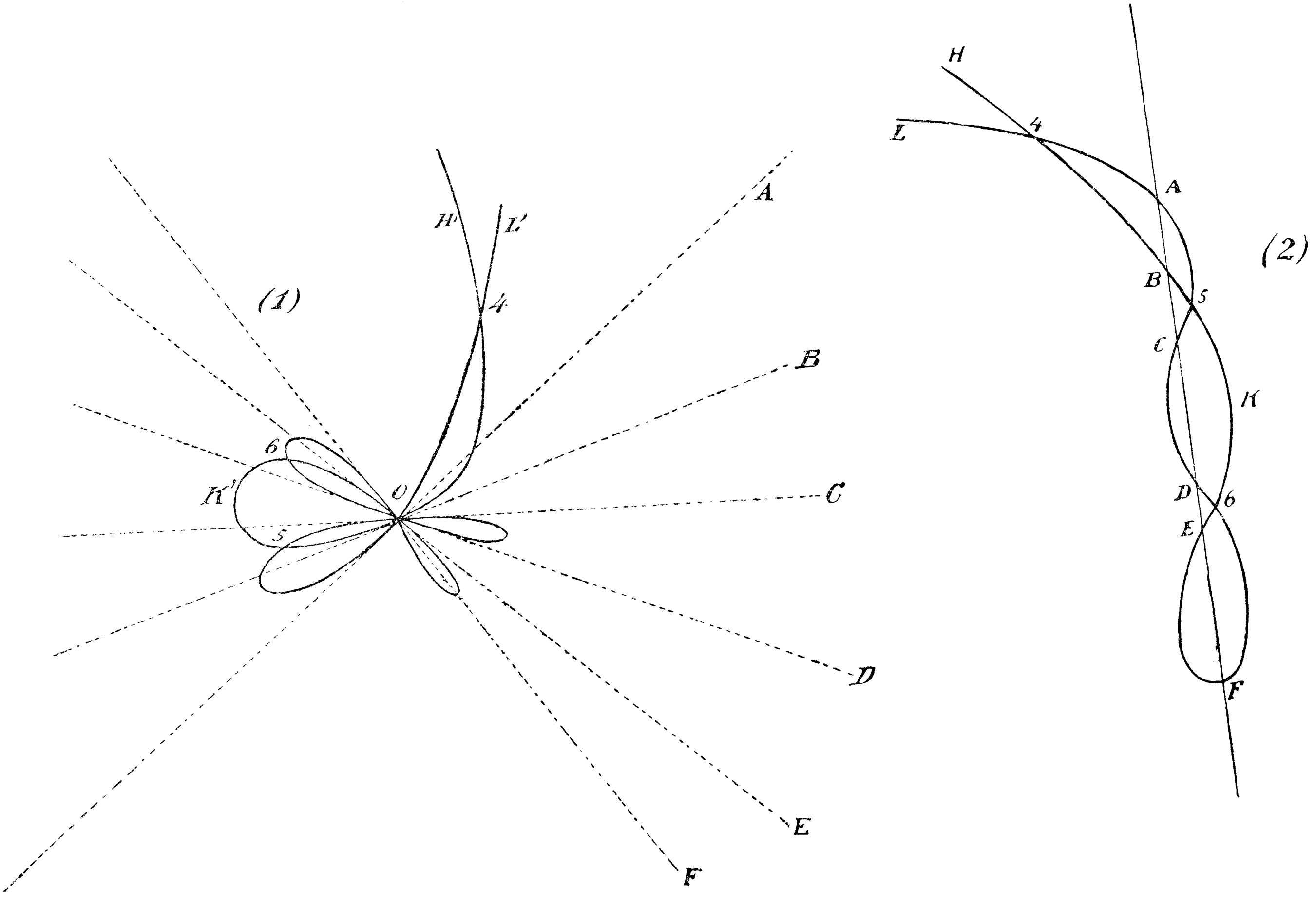

FIG. 26.

The sextic point giving 15 nodes, with 5 loops, we have

$$\delta + \kappa = 15 + 6 = 21, \quad \kappa = 5,$$

the numbers given in the paper referred to. Displacing slightly the components of the sextic point, Fig. (27), the course of the curve is made more distinct. The appearance presented is that of a ramphoid cusp.

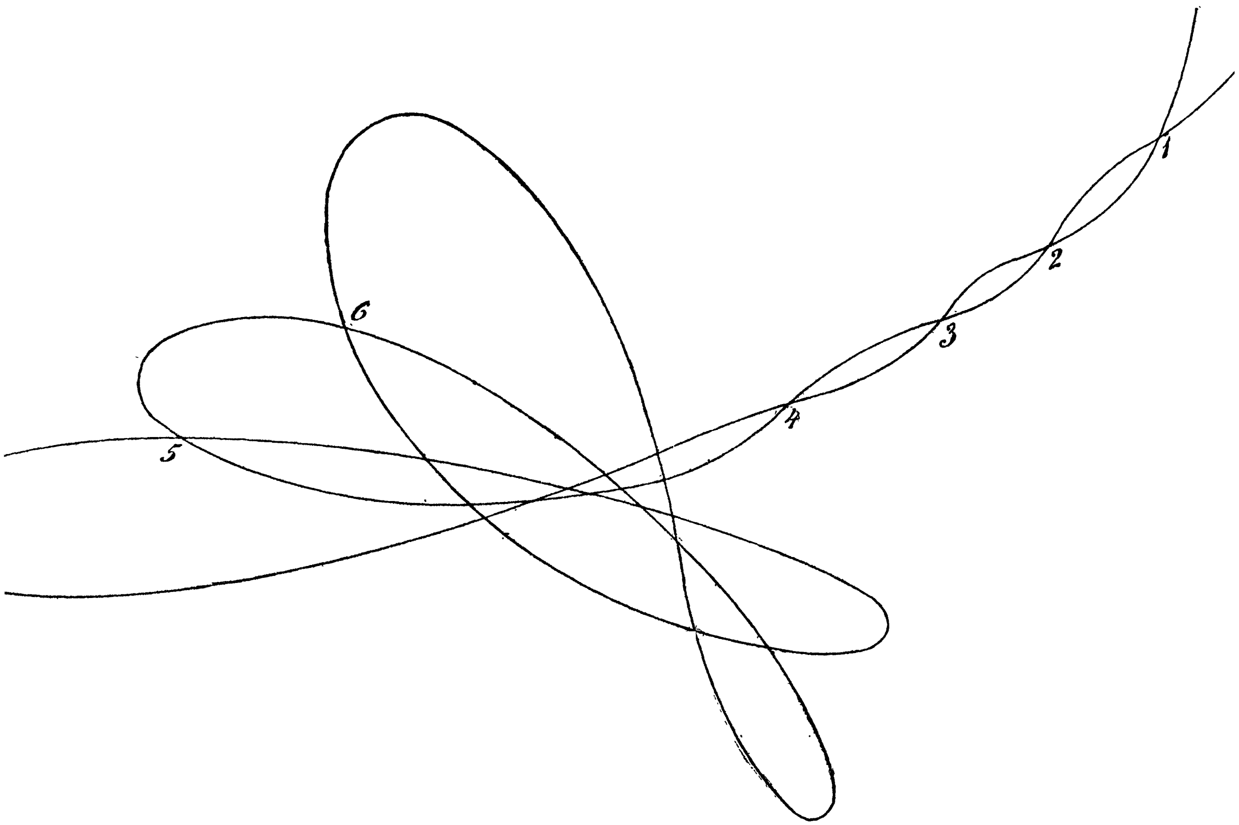

FIG. 27.

18. Applying the process to $y^m = x^p$ (p prime to m and $> m$) (Halphen, "Mémoire sur les points singuliers des courbes algébriques planes," Article II) the first inverse is

$$y_1^m = x_1^{p-m},$$

the second is

$$y_2^m = x_2^{p-2m},$$

and so on, till the exponent of x is less than the exponent of y.

Let

$$\left.\begin{aligned} p &= mk_1 + q, \\ m &= qk_2 + r, \\ q &= rk_3 + s, \text{ etc.} \end{aligned}\right\} \tag{1}$$

Reducing to the form $y^m = x^q$, $q < m$, we see that the base is now the tangent, and meets the curve (the k_1^{th} inverse) in m points. There are therefore in the original $m - 1$ cusps, i. e. $\varkappa = m - 1$. Further, we have resolved the point into k_1 consecutive m-points, together with the dps that are given by

$$y^m = x^q (q < m).$$

We have then, if $\underset{m,p}{\delta}$ denotes the number of nodes in the singularity $y^m = x^p (p > m)$,

$$\underset{m,p}{\delta} = k_1 \times \frac{m(m-1)}{2} + \underset{q,m}{\delta}.$$

This gives us

$$\begin{aligned}\underset{m,p}{\delta} &= k_1 \frac{m(m-1)}{2} + k_2 \frac{q(q-1)}{2} + k_3 \frac{r(r-1)}{2} + \dots\\ &= \tfrac{1}{2}\{(p-q)(m-1) + (m-r)(q-1) + (q-s)(r-1) + \dots\}\end{aligned}$$

from formulæ (1), in which the last of the quantities p, m, q, ... is $= 1$.

$$\begin{aligned}\underset{m,p}{\delta} &= \tfrac{1}{2}\left\{\begin{matrix}(p-q)m + (m-r)q + (q-s)r + \dots \\ -p + q - m + r - q + s \dots + 1\end{matrix}\right\}\\ &= \tfrac{1}{2}\{pm - p - m + 1\}\\ &= \frac{(p-1)(m-1)}{2}.\end{aligned}$$

We get therefore, for the point-components of the singularity $y^m = x^p$,

$$\varkappa = m - 1,\ \delta + \varkappa = \frac{(p-1)(m-1)}{2} \quad \text{(Halphen.)}$$

As was explained in §8, the process does not ordinarily enable us to count the inflexions, though in certain cases they can be detected by means of the tangents from O. In each inversion used in the present example, the tangent is either the original tangent or the base line; and consequently, there being only one complete branch, the inflexions can be counted. The number given by Brill's formulæ (Math. Ann. XVI, 400), $p - m - 1$, presents itself as follows:

The first inverse is $y_1^m = x_1^{p-m}$.

(i) If now $p - m < m$, so that $x_1 = 0$ is the tangent, this singularity contains $p - m - 1$ cusps; each of these indicates an evanescent loop; O *not* being on the coincident tangents, there is from O, to each of these loops, one tangent; these tangents from O give inflexions on the original, in number $p - m - 1$; and this being the only way in which inflexions can occur in the singularity considered, we have for the total number $p - m - 1$.

(ii) If $p - m > m$, $y_1 = 0$ is the tangent, and meets the inverse in $p - m$ points; there being only one branch, this gives $p - m - 1$ contacts, and therefore in the original $p - m - 1$ inflexions.

Finally, knowing that the equation of the reciprocal of the curve $y^m = x^p$ is of the same form, and that the reciprocal singularity contains $p - m - 1$ cusps, $m - 1$ inflexions, we see that the reciprocal singularity has an equation $u^{p-m} = v^p$; and therefore we have

$$\tau + \iota = \frac{(p-1)(p-m-1)}{2}, \quad \iota = p - m - 1.$$

Thus e. g. $x^5y^3 = z^8$.

The 3 . 8 point gives $\delta + \varkappa = 7, \quad \varkappa = 2,$

$\iota = 4, \quad \tau + \iota = 14,$

i. e. $\delta = 5$, $\varkappa = 2$, $\tau = 10$, $\iota = 4$.

The 5 . 8 point gives $\delta + \varkappa = 14, \quad \varkappa = 4,$

$\iota = 2, \quad \tau + \iota = 7,$

i. e. $\delta = 10$, $\varkappa = 4$, $\tau = 5$, $\iota = 2$.

19. In the case of a singularity with coincident tangents, the danger in separating the tangents is that the neighboring multiple points may be destroyed. But these are indicated by multiple points on the base line; we can therefore separate the tangents, provided that, on the inverse, no multiple points are thereby lost in the immediate neighborhood of the base line; and we can substitute nodes and loops for the cusps, by displacing the base line so that there may no longer be contact.

Take for example the curve (Cramer)

$$(y - x^2)^3 - x^3y^2 = 0, \tag{1}$$

this has two consecutive triple points, involving two cusps. The first inverse is

$$(y_1 - x_1)^3 - x_1^2y_1^2 = 0, \tag{2}$$

which by the substitution $y_1 = x_1 + u_1$, becomes

$$u_1^3 - x_1^2(x_1 + u_1)^2 = 0. \tag{3}$$

The inverse of this, obtained by $x_1 = x_2$, $u_1 = x_2y_2$, is

$$y_2^3 - x_2(1 + y_2)^2 = 0. \tag{4}$$

The curve (4) has an inflexion at x_2y_2; no multiple points are lost if we resolve the inflexion in the ordinary way, separating the three consecutive points by writing $y_2(y_2-\lambda)(y_2+\lambda)$ instead of y_2^3. We get then, instead of (4),

$$(y_2^2-\lambda^2)y_2-x_2(1+y_2)^2=0; \tag{4'}$$

instead of (3) we have

$$(u_1^2-\lambda^2x_1^2)u_1-x_1^2(x_1+u_1)^2=0; \tag{3'}$$

instead of (2), $$\{(y_1-x_1)^2-\lambda^2x_1^2\}(y_1-x_1)-x_1^2y_1^2=0. \tag{2'}$$

In passing to (1), $x_1=0$ is the base line, and it passes through the triple point at x_1y_1; take instead as base line $x_1=\mu$; i. e. change the x_1 into $x_1+\mu$; and we get

$$\{(y_1-x_1-\mu)^2-\lambda^2(x_1+\mu)^2\}\{y_1-x_1-\mu\}-\{x_1+\mu\}^2y_1^2=0,$$

which gives us, instead of (1),

$$\{(y-x^2-\mu x)^2-\lambda^2x^2(x+\mu)^2\}\{y-x^2-\mu x\}-xy^2(x+\mu)^2=0. \tag{1'}$$

This has the two triple points separated, they being now at $0\,.\,0$ and at $-\mu\,.\,0$. The tangents at each are separated, and the cusps represented by loops.

If however we take $y^3=x^5$, which gives as the first inverse $y_1^3=x_1^2$, we cannot separate the *three* points on the base line; for by so doing we should destroy a dp. But we may substitute for y_1^3, $y_1^2(y_1+\lambda)$, which reduces the cusp to a node and loop; and then take a new base line which does not pass through the node, as in the last example.

BRYN MAWR COLLEGE, Bryn Mawr, Pa., June, 1892.